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Solution structure of hyperbolic heat-conduction equation

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Abstract

The contributions of the initial temperature distribution φ and source disturbance f to the temperature field T in the hyperbolic heat conduction are related to that of the initial rate of temperature change ψ. This uncovers the structure of the temperature field and significantly simplifies the development of solutions of hyperbolic heat-conduction equations.

1. Introduction

Heat waves in various conduction domains are generated by initial, boundary and source disturbances, and are controlled by the hyperbolic heat conduction equation. Mathematically, they are characterized by solutions of initial-boundary value problems in the form of

$$\begin{cases} \dfrac{T_t}{\tau_0} + T_{tt} = a^2 \Delta T + f(M, t); \quad \Omega \times (0, +\infty) \\ g(T, T_{\mathrm{n}})\,|_{\partial\Omega} = h(t); \quad (0, +\infty) \\ T(M, 0) = \varphi(M), \quad T_t(M, 0) = \psi(M); \quad \Omega \end{cases} \tag{1}$$

if all thermophysical material properties are assumed to be constant. Here t is the time, T the temperature, τ_0 the thermal relaxation time, $a^2 = \alpha/\tau_0$ with α as the thermal diffusivity, M denotes a point in the space domain Ω with the boundary $\partial\Omega$, Δ the Laplacian, f, g, h, φ and ψ are functions, T_{n} the normal derivative of T, $T_t = \partial T/\partial t$, and $T_{tt} = \partial^2 T/\partial t^2$, etc. This, with various applications of heat waves in various fields, has given rise to a considerable body of literature on the solution of (1), mainly for a one-dimensional case with a linear function g. Refs. [1–11] obtained either analytical or numerical solutions of (1) for some g and h to examine features of heat waves due to boundary disturbances. Wilhelm and Choi [12] developed a solution to study heat waves due to an initial disturbance φ of a delta-like temperature distribution centered on a line. Works in Refs. [13–17] devoted to solutions of (1) to observe characteristics of heat waves due to some source disturbances f. The features of heat waves revealed include the sharp wavefront, thermal shock, thermal resonance, reflection, refraction and transmission, etc. The readers are referred to Refs. [18–24] for some excellent reviews and discussions of this important topic.

The solution domain Ω considered in previous works is mostly restricted to infinite, semi-infinite, and one-dimensional. The known solutions of (1) are for some specific Ω, f, g, h, φ and ψ. The analytical method used is mainly integration transformation including Fourier and Laplace transformations. When a finite domain is considered, the analysis becomes an intricate matter as disturbances travel as a wave while dissipating and reflecting off the boundaries. Due to the complicated reflection and interaction of waves, multi-dimensional heat waves would contain richer features. The interaction among waves due to various boundary, initial and source disturbances would also

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Nomenclature

a_n	coefficient
a^2	α/τ_0
b_n	coefficient
c_1	constant
c_2	constant
$c_1^{(n)}$	coefficient
$c_2^{(n)}$	coefficient
f	function
g	function
h	function
l	thickness
L	linear function
M	point
t	time
T, T_1, T_2, T_3	temperature
T_n	normal derivative of temperature T
W	solution function
x	coordinate

Greek symbols

α	thermal diffusivity
γ_n	variable
Δ	Laplacian
τ	variable
τ_0	thermal relaxation time
φ	function
ψ	function
Ω	space domain
$\partial\Omega$	boundary of Ω

lead to much richer features. To develop solutions of (1) for various Ω, f, g, h, φ and ψ is thus of considerable importance to reveal new features of heat waves which find applications in various fields. This requires, however, large amounts of effort. The motivation for the present work comes from the desire to reveal the solution structure of (1) to reduce such effort. In particular, we develop two solution structure theorems for the case of a linear function g. Note that commonly-used Dirichlet, Neumann and Robin boundary conditions are the special cases of a linear function g.

2. Solution structure

As the effect of h can be transformed to the effect of source term through a homogenization of boundary conditions, we only need to search the solution structure of problem

$$\begin{cases} \dfrac{T_t}{\tau_0} + T_{tt} = a^2\Delta T + f(M, t); & \Omega \times (0, +\infty) \\ L(T, T_n)\,|_{\partial\Omega} = 0; & (0, +\infty) \\ T(M, 0) = \varphi(M), \quad T_t(M, 0) = \psi(M); & \Omega \end{cases} \tag{2}$$

where $L(T, T_n)$ represents linear functions of T and T_n, $L(T, T)|_{\partial\Omega} = 0$ denotes homogeneous boundary conditions.

Applying the principle of superposition to (2) yields

$$T(M, t) = T_1(M, t) + T_2(M, t) + T_3(M, t), \tag{3}$$

where $T_1(M, t)$, $T_2(M, t)$ and $T_3(M, t)$ are the solutions of

$$\begin{cases} \dfrac{T_t}{\tau_0} + T_{tt} = a^2\Delta T; & \Omega \times (0, +\infty) \\ L(T, T_n)\,|_{\partial\Omega} = 0; & (0, +\infty) \\ T(M, 0) = 0, \quad T_t(M, 0) = \psi(M); & \Omega, \end{cases} \tag{4}$$

$$\begin{cases} \dfrac{T_t}{\tau_0} + T_{tt} = a^2\Delta T; & \Omega \times (0, +\infty) \\ L(T, T_n)\,|_{\partial\Omega} = 0; & (0, +\infty) \\ T(M, 0) = \varphi(M), \quad T_t(M, 0) = 0; & \Omega, \end{cases} \tag{5}$$

and

$$\begin{cases} \dfrac{T_t}{\tau_0} + T_{tt} = a^2\Delta T + f(M, t); & \Omega \times (0, +\infty) \\ L(T, T_n)\,|_{\partial\Omega} = 0; & (0, +\infty) \\ T(M, 0) = 0, \quad T_t(M, 0) = 0; & \Omega, \end{cases} \tag{6}$$

respectively. Now we proceed to prove two theorems relating T_2 and T_3 to T_1. In the process of deriving the two theorems, a commonly-used assumption is made that the order of differentiation is interchangeable for some high-order partial derivatives of T with respect to the time and space coordinates. While the continuity of the associated high-order partial derivatives forms the sufficient condition for such interchange, it is not the necessary condition. Therefore, the interchange of the order of differentiation could still be valid even in the wave-front region where some high-order partial derivatives of T could be discontinuous (the dissipating or damping feature of hyperbolic heat-conduction equations would hinder the appearance of such discontinuity). As both the necessary and sufficient condition is not available in mathematics, it appears not possible, at the present, to state what are the conditions that T should possess in order to be able to interchange the order of differentiation. However, this assumption appears valid for solutions of hyperbolic heat-conduction equations because the two theorems give rise to the same solution as that obtained by the Fourier method (see Section 3).

2.1. Theorem 1

Theorem 1. Let $W_\psi(M, t)$ denote the solution of (4). The solution of (5) can be written as

$$T_2(M,t)=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t). \tag{7}$$

As $W_\psi(M, t)$ is the solution of (4), we have

$$\begin{cases}\dfrac{1}{\tau_0}\dfrac{\partial}{\partial t}W_\varphi(M,t)+\dfrac{\partial^2}{\partial t^2}W_\varphi(M,t)=a^2\Delta W_\varphi(M,t); \\ \quad \Omega\times(0,\ +\infty) \\ L\left[W_\varphi(M,t),\dfrac{\partial}{\partial n}W_\varphi(M,t)\right]\Big|_{\partial\Omega}=0; \quad (0,\ +\infty) \\ W_\varphi(M,0)=0, \quad \dfrac{\partial}{\partial t}W_\varphi(M,0)=\varphi(M); \quad \Omega.\end{cases} \tag{8}$$

Hence,

$$\frac{1}{\tau_0}\frac{\partial}{\partial t}T_2+\frac{\partial^2}{\partial t^2}T_2-a^2\Delta T_2$$

$$=\frac{1}{\tau_0}\frac{\partial}{\partial t}\left[\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t)\right]$$

$$+\frac{\partial^2}{\partial t^2}\left[\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t)\right]$$

$$-a^2\Delta\left[\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t)\right]$$

$$=\frac{1}{\tau_0}\left[\frac{1}{\tau_0}\frac{\partial}{\partial t}W_\varphi(M,t)+\frac{\partial^2}{\partial t^2}W_\varphi(M,t)-a^2\Delta W_\varphi(M,t)\right]$$

$$+\frac{\partial}{\partial t}\left[\frac{1}{\tau_0}\frac{\partial}{\partial t}W_\varphi(M,t)+\frac{\partial^2}{\partial t^2}W_\varphi(M,t)-a^2\Delta W_\varphi(M,t)\right]$$

$$=0,$$

which indicates that T_2 in (7) satisfies the equation of (5).

Also,

$$L\left(T_2,\frac{\partial}{\partial n}T_2\right)=L\left\{\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t),\right.$$

$$\left.\frac{\partial}{\partial n}\left[\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t)\right]\right\}$$

$$=\frac{1}{\tau_0}L\left[W_\varphi(M,t),\frac{\partial}{\partial n}W_\varphi(M,t)\right]$$

$$+\frac{\partial}{\partial t}L\left[W_\varphi(M,t),\frac{\partial}{\partial n}W_\varphi(M,t)\right],$$

and

$$L\left(T_2,\frac{\partial}{\partial n}T_2\right)\Big|_{\partial\Omega}=\frac{1}{\tau_0}L\left[W_\varphi(M,t),\frac{\partial}{\partial n}W_\varphi(M,t)\right]\Big|_{\partial\Omega}$$

$$+\frac{\partial}{\partial t}L\left[W_\varphi(M,t),\frac{\partial}{\partial n}W_\varphi(M,t)\right]\Big|_{\partial\Omega}$$

$$=\frac{1}{\tau_0}L\left[W_\varphi(M,t),\frac{\partial}{\partial n}W_\varphi(M,t)\right]\Big|_{\partial\Omega}$$

$$+\frac{\partial}{\partial t}\left\{L\left[W_\varphi(M,t),\frac{\partial}{\partial n}W_\varphi(M,t)\right]\Big|_{\partial\Omega}\right\}=0,$$

in which (8) has been used. This indicates that the T_2 in (7) satisfies the boundary condition of (5).

Finally,

$$T_2(M,0)=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t)\mid_{t=0}$$

$$=\frac{1}{\tau_0}W_\varphi(M,0)+\frac{\partial}{\partial t}W_\varphi(M,0)=\varphi(M),$$

by (8). Also,

$$\frac{\partial}{\partial t}T_2(M,0)=\frac{\partial}{\partial t}\left[\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t)\right]\Big|_{t=0}$$

$$=\left[\frac{1}{\tau_0}\frac{\partial}{\partial t}W_\varphi(M,t)+\frac{\partial^2}{\partial t^2}W_\varphi(M,t)\right]\Big|_{t=0}$$

$$=a^2\Delta W_\varphi(M,t)\mid_{t=0}=a^2\Delta[W_\varphi(M,t)\mid_{t=0}=0,$$

by (8). Therefore, the T_2 in (7) also satisfies the initial conditions of (5).

2.2. Theorem 2

Theorem 2. Let $W_\psi(M, t)$ denote the solution of (4).

The solution of (6) can be written as

$$T_3(M,t)=\int_0^t W_{f_\tau}(M,t-\tau)\mathrm{d}\tau, \tag{9}$$

where

$$f_\tau=f(M,\tau).$$

As $W_\psi(M,t)$ is the solution of (4), we have

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial}{\partial t}W_{f_\tau}(M,t-\tau)+\dfrac{\partial^2}{\partial t^2}W_{f_\tau}(M,t-\tau) \\ =a^2\Delta W_{f_\tau}(M,t-\tau);\quad \Omega\times(0,+\infty) \\ L\left[W_{f_\tau}(M,t-\tau),\dfrac{\partial}{\partial n}W_{f_\tau}(M,t-\tau)\right]\Bigg|_{\partial\Omega} \\ =0;\quad (0,+\infty) \\ W_{f_\tau}(M,t-\tau)\mid_{t=\tau}=0,\quad \dfrac{\partial}{\partial t}W_{f_\tau}(M,t-\tau)\mid_{t=\tau} \\ =f(M,\tau);\quad \Omega. \end{cases} \tag{10}$$

Therefore,

$$\begin{aligned} \frac{1}{\tau_0}\frac{\partial}{\partial t}T_3+\frac{\partial^2}{\partial t^2}T_3-a^2\Delta T_3 &=\frac{1}{\tau_0}\frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M,t-\tau)\mathrm{d}\tau \\ &+\frac{\partial^2}{\partial t^2}\int_0^t W_{f_\tau}(M,t-\tau)\mathrm{d}\tau-a^2\Delta\int_0^t W_{f_\tau}(M,t-\tau)\mathrm{d}\tau \\ &=\frac{1}{\tau_0}\left[\int_0^t\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}\mathrm{d}\tau+W_{f_\tau}(M,t-\tau)\Bigg|_{\tau=t}\right] \\ &+\frac{\partial}{\partial t}\left[\int_0^t\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}\mathrm{d}\tau+W_{f_\tau}(M,t-\tau)\Bigg|_{\tau=t}\right] \\ &-a^2\Delta\int_0^t W_{f_\tau}(M,t-\tau)\mathrm{d}\tau \\ &=\frac{1}{\tau_0}\int_0^t\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}\mathrm{d}\tau+\int_0^t\frac{\partial^2 W_{f_\tau}(M,t-\tau)}{\partial t^2}\mathrm{d}\tau \\ &+\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}\Bigg|_{\tau=t}-a^2\Delta\int_0^t W_{f_\tau}(M,t-\tau)\mathrm{d}\tau \\ &=\int_0^t\left[\frac{1}{\tau_0}\frac{\partial}{\partial t}W_{f_\tau}(M,t-\tau)+\frac{\partial^2}{\partial t^2}W_{f_\tau}(M,t-\tau)\right. \\ &\left.-a^2\Delta W_{f_\tau}(M,t-\tau)\right]\mathrm{d}\tau+f(M,t)=f(M,t), \end{aligned}$$

which indicates that the T_3 in (9) satisfies the equation of (6).

Also,

$$\begin{aligned} L\left(T_3,\frac{\partial}{\partial n}T_3\right)\Bigg|_{\partial\Omega} &=L\left[\int_0^t W_{f_\tau}(M,t-\tau)\mathrm{d}\tau,\right. \\ &\left.\frac{\partial}{\partial n}\int_0^t W_{f_\tau}(M,t-\tau)\mathrm{d}\tau\right]\Bigg|_{\partial\Omega}=\int_0^t L\left[W_{f_\tau}(M,t-\tau),\right. \\ &\left.\frac{\partial}{\partial n}W_{f_\tau}(M,t-\tau)\right]\Bigg|_{\partial\Omega}\mathrm{d}\tau=0, \end{aligned}$$

in which (10) has been used. Therefore, the T_3 in (9) satisfies the boundary condition of (6).

Finally,

$$T_3(M,0)=\int_0^0 W_{f_\tau}(M,t-\tau)\mathrm{d}\tau=0,$$

and

$$\begin{aligned} \frac{\partial}{\partial t}T_3(M,t)\mid_{t=0} &=\left[\int_0^t\frac{\partial}{\partial t}W_{f_\tau}(M,t-\tau)\mathrm{d}\tau\right. \\ &\left.+W_{f_\tau}(M,t-\tau)\mid_{\tau=t}\right]\Bigg|_{t=0}=0, \end{aligned}$$

by (10). Therefore, the T_3 in (9) also satisfies the initial conditions of (6).

3. Applications

Two theorems proven in this paper relate the solution of (2) to that of (4) which is much easier to solve. They are also valid for Cauchy problems (initial value problems). To illustrate their applications and compare with other methods, we use both the Fourier method and two theorems to solve the one-dimensional initial-boundary value problem

$$\begin{cases} \dfrac{T_t}{\tau_0}+T_{tt}=a^2T_{xx}+f(x,t);\quad (0,l)\times(0,+\infty) \\ T(0,t)=T(l,t)=0;\quad (0,+\infty) \\ T(x,0)=\varphi(x),\quad T_t(x,0)=\psi(x);\quad (0,l), \end{cases} \tag{11}$$

whose solution represents the temperature distribution in an infinitely-wide solid slab of thickness l.

3.1. The solution by the Fourier method

3.1.1. The ψ-initiated solution: $W_\psi(x,t)$

For the problem

$$\begin{cases} \dfrac{T_t}{\tau_0} + T_{tt} = a^2 T_{xx}; \quad (0, l) \times (0, +\infty) \\ T(0, t) = T(l, t) = 0; \quad (0, +\infty) \\ T(x, 0) = 0, \quad T_t(x, 0) = \psi(x); \quad (0, l), \end{cases} \tag{12}$$

which governs the ψ-contribution. By taking the boundary conditions into account, let

$$T(x, t) = \sum_{n=1}^{+\infty} T_n(t) \sin \frac{n\pi x}{l}. \tag{13}$$

Substituting this into the equation of (12) yields

$$\frac{\mathrm{d}^2 T_n}{\mathrm{d}t^2} + \frac{1}{\tau_0} \frac{\mathrm{d}T_n}{\mathrm{d}t} + \left(\frac{n\pi a}{l}\right)^2 T_n = 0. \tag{14}$$

The solution of (14) can be easily found as

$$T_n(t) = \mathrm{e}^{-(t/2\tau_0)}(a_n \cos \gamma_n t + b_n \underline{\sin} \gamma_n t), \tag{15}$$

where a_n and b_n are constants to be determined by the initial conditions,

$$\gamma_n \equiv \frac{1}{2\tau_0} \sqrt{4\tau_0^2 \left(\frac{n\pi a}{l}\right)^2 - 1}, \tag{16}$$

and

$$\underline{\sin} \gamma_n t = \begin{cases} \sin \gamma_n t & \text{if } \gamma_n \neq 0 \\ t & \text{if } \gamma_n = 0. \end{cases} \tag{17}$$

We thus have

$$T(x, t) = \sum_{n=1}^{+\infty} \mathrm{e}^{-(t/2\tau_0)}(a_n \cos \gamma_n t + b_n \underline{\sin} \gamma_n t) \sin \frac{n\pi x}{l}. \tag{18}$$

Applying the first initial condition $T(x, 0) = 0$ leads to

$$\sum_{n=1}^{+\infty} a_n \sin \frac{n\pi x}{l} = 0, \tag{19}$$

which requires

$$a_n = 0, \quad n = 1, 2, \ldots. \tag{20}$$

Hence,

$$T(x, t) = \sum_{n=1}^{+\infty} \mathrm{e}^{-(t/2\tau_0)} b_n \underline{\sin} \gamma_n t \sin \frac{n\pi x}{l}, \tag{21}$$

and

$$T_t(x, t) = \sum_{n=1}^{+\infty} b_n \left(-\frac{1}{2\tau_0} \underline{\sin} \gamma_n t + \underline{\gamma_n} \cos \gamma_n t\right) \mathrm{e}^{-(t/2\tau_0)} \sin \frac{n\pi x}{l}, \tag{22}$$

where

$$\underline{\gamma_n} = \begin{cases} \gamma_n & \text{if } \gamma_n \neq 0 \\ 1 & \text{if } \gamma_n = 0. \end{cases} \tag{23}$$

Applying the second initial condition $T_t(x, 0) = \psi(x)$ yields

$$\sum_{n=1}^{+\infty} b_n \underline{\gamma_n} \sin \frac{n\pi x}{l} = \psi(x), \tag{24}$$

which requires

$$b_n = \frac{2}{l\underline{\gamma_n}} \int_0^l \psi(\xi) \sin \frac{n\pi \xi}{l} \mathrm{d}\xi, \quad n = 1, 2, \ldots. \tag{25}$$

Finally, we have

$$T(x, t) = \sum_{n=1}^{+\infty} \left(\frac{2}{l\underline{\gamma_n}} \int_0^l \psi(\xi) \sin \frac{n\pi \xi}{l} \mathrm{d}\xi\right) \mathrm{e}^{-(t/2\tau_0)} \underline{\sin} \gamma_n t \sin \frac{n\pi x}{l}. \tag{26}$$

3.1.2. The φ-initiated solution: $T_2(x, t)$

For the problem

$$\begin{cases} \dfrac{T_t}{\tau_0} + T_{tt} = a^2 T_{xx}; \quad (0, l) \times (0, +\infty) \\ T(0, t) = T(l, t) = 0; \quad (0, +\infty) \\ T(x, 0) = \varphi(x), \quad T_t(x, 0) = 0; \quad (0, l), \end{cases} \tag{27}$$

which governs the φ-contribution, let

$$T(x, t) = \sum_{n=1}^{+\infty} T_n(t) \sin \frac{n\pi x}{l}. \tag{28}$$

Such a solution satisfies two boundary conditions in (27). A substitution of (28) into the equation in (27) leads to (15) again for $T_n(t)$. Hence,

$$T(x, t) = \sum_{n=1}^{+\infty} \mathrm{e}^{-(t/2\tau_0)}(a_n \cos \gamma_n t + b_n \underline{\sin} \gamma_n t) \sin \frac{n\pi x}{l}. \tag{29}$$

Applying the first initial condition $T(x, 0) = \varphi(x)$ leads to

$$\sum_{n=1}^{+\infty} a_n \sin\frac{n\pi x}{l} = \varphi(x), \tag{30}$$

which requires

$$a_n = \frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{n\pi\xi}{l}\mathrm{d}\xi, \quad n = 1, 2, \ldots. \tag{31}$$

Also,

$$\begin{aligned} T_t(x, t) = \sum_{n=1}^{+\infty}\Big\{ &-\frac{1}{2\tau_0}\mathrm{e}^{-(t/2\tau_0)}(a_n \cos\gamma_n t \\ &+ b_n \underline{\sin}\,\gamma_n t) + \mathrm{e}^{-(t/2\tau_0)}(-a_n\gamma_n \sin\gamma_n t \\ &+ b_n\underline{\gamma_n}\cos\gamma_n t)\Big\}\sin\frac{n\pi x}{l}. \end{aligned} \tag{32}$$

Applying the second initial condition $T_t(x, 0) = 0$ yields

$$\sum_{n=1}^{+\infty}\left(-\frac{1}{2\tau_0}a_n + b_n\underline{\gamma_n}\right)\sin\frac{n\pi x}{l} = 0, \tag{33}$$

which requires

$$b_n = \frac{a_n}{2\tau_0\underline{\gamma_n}}. \tag{34}$$

Finally, we have

$$\begin{aligned} T_2(x, t) = &\sum_{n=1}^{+\infty}\left(\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{n\pi\xi}{l}\mathrm{d}\xi\right)\mathrm{e}^{-(t/2\tau_0)}\cos \\ &\gamma_n t\sin\frac{n\pi x}{l} + \sum_{n=1}^{+\infty}\left(\frac{1}{l\underline{\gamma_n}\tau_0}\int_0^l\varphi(\xi)\sin\frac{n\pi\xi}{l}\mathrm{d}\xi\right) \\ &\mathrm{e}^{-(t/2\tau_0)}\,\underline{\sin}\,\gamma_n t\sin\frac{n\pi x}{l}. \end{aligned} \tag{35}$$

3.1.3. The f-initiated solution: $T_3(x, t)$

Consider the problem

$$\begin{cases} \dfrac{T_t}{\tau_0} + T_{tt} = a^2T_{xx} + f(x, t); \quad (0, l)\times(0, +\infty) \\ T(0, t) = T(l, t) = 0; \quad (0, +\infty) \\ T(x, 0) = 0, \quad T_t(x, 0) = 0; \quad (0, l), \end{cases} \tag{36}$$

which governs the contribution of the source term f. Using the Fourier sine series to express the f as

$$f(x, t) = \sum_{n=1}^{+\infty} f_n(t)\sin\frac{n\pi x}{l}, \tag{37}$$

where

$$f_n(t) = \frac{2}{l}\int_0^l f(\xi, t)\sin\frac{n\pi\xi}{l}\mathrm{d}\xi. \tag{38}$$

Let

$$T(x, t) = \sum_{n=1}^{+\infty} T_n(t)\sin\frac{n\pi x}{l}. \tag{39}$$

Such a solution satisfies two boundary conditions in (36). A substitution of (37) and (39) into (36) yields

$$\frac{T'_{\mathrm{n}}(t)}{\tau_0} + T''_{\mathrm{n}}(t) + \left(\frac{n\pi a}{l}\right)^2 T_{\mathrm{n}}(t) = f_n(t), \tag{40}$$

and

$$T_{\mathrm{n}}(0) = T'_{\mathrm{n}}(0) = 0. \tag{41}$$

The solution of (40) can be, by (15) and the method of variable coefficients, written as

$$T_{\mathrm{n}}(t) = c_1^{(n)}(t)y_1 + c_2^{(n)}y_2, \tag{42}$$

where

$$y_1 = \mathrm{e}^{-(t/2\tau_0)}\cos\gamma_n t, \tag{43}$$

$$y_2 = \mathrm{e}^{-(t/2\tau_0)}\,\underline{\sin}\,\gamma_n t, \tag{44}$$

$$c_1^{(n)}(t) = \int -\frac{f_n(t)y_2}{\underline{\Delta}}\mathrm{d}t + c_1, \tag{45}$$

$$c_2^{(n)}(t) = \int\frac{f_n(t)y_1}{\underline{\Delta}}\mathrm{d}t + c_2. \tag{46}$$

Here c_1 and c_2 are constants, and

$$\underline{\Delta} = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \underline{\gamma_n}\,\mathrm{e}^{-(t/\tau_0)}. \tag{47}$$

By $T_{\mathrm{n}}(0) = 0$, we have

$$c_1^{(n)}(0) = 0. \tag{48}$$

Therefore,

$$\begin{aligned} c_1^{(n)}(t) &= \int -\frac{f_n(t)y_2}{\underline{\Delta}}\mathrm{d}t + c_1 \\ &= \int -\frac{f_n(t)\,\mathrm{e}^{-(t/2\tau_0)}\,\underline{\sin\gamma_n t}}{\underline{\gamma_n}\,\mathrm{e}^{-(t/\tau_0)}}\mathrm{d}t + c_1 \\ &= -\frac{1}{\underline{\gamma_n}}\int f_n(t)\,\mathrm{e}^{-(t/2\tau_0)}\,\underline{\sin\gamma_n t}\,\mathrm{d}t + c_1 \\ &= -\frac{1}{\underline{\gamma_n}}\int_0^t f_n(\tau)\,\mathrm{e}^{-(\tau/2\tau_0)}\,\underline{\sin\gamma_n\tau}\,\mathrm{d}\tau, \end{aligned} \tag{49}$$

and

$$\begin{aligned} T_n'(t) &= [c_1^{(n)}(t)]'y_1 + c_1^{(n)}(t)y_1' + [c_2^{(n)}(t)]'y_2 + c_2^{(n)}(t)y_2' \\ &= \left(-\frac{1}{\gamma_n} f_n(t)\, e^{-(t/2\tau_0)} \sin\gamma_n t\right) y_1 \\ &\quad + c_1^{(n)}(t)y_1' + [c_2^{(n)}(t)]'y_2 + c_2^{(n)}(t) \\ &\quad e^{-(t/2\tau_0)}\left(-\frac{1}{2\tau_0}\sin\gamma_n t + \gamma_n \cos\gamma_n t\right). \end{aligned}$$

Applying $T_n'(0) = 0$ leads to

$$c_2^{(n)}(0) = 0. \tag{50}$$

Hence,

$$\begin{aligned} c_2^{(n)}(t) &= \int \frac{f_n(t)y_1}{\Delta}\mathrm{d}t + c_2 \\ &= \int \frac{f_n(t)\, e^{-(t/2\tau_0)} \cos\ \gamma_n t}{\gamma_n\, e^{-(t/\tau_0)}}\mathrm{d}t + c_2 = \frac{1}{\gamma_n}\int_0^t f_n(\tau) \\ &\quad e^{(\tau/2\tau_0)} \cos\ \gamma_n\tau\, \mathrm{d}\tau. \end{aligned} \tag{51}$$

Substituting (49) and (51) into (42) yields

$$\begin{aligned} T_n(t) &= e^{-(t/2\tau_0)} \cos\ \gamma_n t\left(-\frac{1}{\gamma_n}\int_0^t f_n(\tau)\, e^{(\tau/2\tau_0)} \sin \gamma_n\tau\, \mathrm{d}\tau\right) + e^{-(t/2\tau_0)} \sin\gamma_n t\left(\frac{1}{\gamma_n}\int_0^t f_n(\tau)\, e^{(\tau/2\tau_0)} \cos \gamma_n\tau\, \mathrm{d}\tau\right) \\ &= -\frac{1}{\gamma_n}\int_0^t e^{-[(t-\tau)/2\tau_0]}\frac{1}{2}[\sin \gamma_n(\tau+t) + \sin\gamma_n(\tau-t)]f_n(\tau)\mathrm{d}\tau \\ &\quad + \frac{1}{\gamma_n}\int_0^t e^{-[(t-\tau)/2\tau_0]}\frac{1}{2}[\sin\gamma_n(t+\tau) + \sin\gamma_n(t-\tau)]f_n(\tau)\mathrm{d}\tau = \frac{1}{\gamma_n}\int_0^t e^{-[(t-\tau)/2\tau_0]} \sin \gamma_n(t-\tau)f_n(\tau)\mathrm{d}\tau \\ &= \frac{2}{l\gamma_n}\int_0^l \mathrm{d}\xi\int_0^t e^{-[(t-\tau)/2\tau_0]} \sin \gamma_n(t-\tau)\sin\frac{n\pi\xi}{l}f(\xi,\tau)\mathrm{d}\tau, \end{aligned} \tag{52}$$

in which (38) has been used.

Finally, substituting (52) into (39) leads to

$$T(x,t) = \int_0^l \mathrm{d}\xi\int_0^t G(x,\xi,t-\tau)f(\xi,\tau)\mathrm{d}\tau, \tag{53}$$

in which,

$$\begin{aligned} &G(x,\xi,t-\tau) \\ &= \frac{2}{l}\sum_{n=1}^{+\infty}\frac{1}{\gamma_n}\, e^{-[(t-\tau)/2\tau_0]}\ \sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}\sin\ \gamma_n(t-\tau), \end{aligned} \tag{54}$$

is termed as the fundamental solution of (36). When

$$f(x,t) = \delta(x-x_0, t-t_0),$$

$$T(x,t) = G(x, x_0, t-t_0).$$

3.2. The solution by two theorems

3.2.1. The φ-initiated solution: $T_2(x, t)$

By applying Theorem 1 to this one-dimensional problem,

$$T_2(x,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi(x,t). \tag{55}$$

Using the W in (26), we have

$$\begin{aligned} T_2(x,t) &= \sum_{n=1}^{+\infty} e^{-(t/2\tau_0)} \sin \gamma_n t\ \sin\frac{n\pi x}{l}\left(\frac{2}{l\gamma_n\tau_0}\int_0^l\varphi(\xi)\sin\frac{n\pi\xi}{l}\mathrm{d}\xi\right) \\ &\quad + \sum_{n=1}^{+\infty} e^{-(t/2\tau_0)}\left(-\frac{1}{2\tau_0}\sin\gamma_n t + \gamma_n\cos\gamma_n t\right)\sin\frac{n\pi x}{l}\left(\frac{2}{l\gamma_n}\int_0^l\varphi(\xi)\sin\frac{n\pi\xi}{l}\mathrm{d}\xi\right) \\ &= \sum_{n=1}^{+\infty}\left(\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{n\pi\xi}{l}\mathrm{d}\xi\right)e^{-(t/2\tau_0)}\cos\gamma_n t\sin\frac{n\pi x}{l} + \sum_{n=1}^{+\infty}\left(\frac{1}{l\gamma_n\tau_0}\int_0^l\varphi(\xi)\sin\frac{n\pi\xi}{l}\mathrm{d}\xi\right) \\ &\quad e^{-(t/2\tau_0)}\sin\gamma_n t\sin\frac{n\pi x}{l}, \end{aligned} \tag{56}$$

which is the same as that obtained by the Fourier method [Eq. (35)].

3.2.2. The f-initiated solution: $T_3(x, t)$

By applying Theorem 2,

$$T_3(x,t) = \int_0^t W_{f_\tau}(x, t-\tau)\mathrm{d}\tau, \tag{57}$$

where

$$f_\tau = f(x, \tau).$$

Using the W in (26), we have

$$T_3(x,t) = \int_0^t \left[\sum_{n=1}^{+\infty} \left(\frac{2}{l\underline{\gamma_n}} \int_0^l f(\xi, \tau) \sin\frac{n\pi\xi}{l}\mathrm{d}\xi \right) \mathrm{e}^{-[(t-\tau)/2\tau_0]}\, \underline{\sin}\, \gamma_n(t-\tau) \sin\frac{n\pi x}{l} \right] \mathrm{d}\tau = \int_0^l \mathrm{d}\xi \int_0^t G(x, \xi, t-\tau) f(\xi, \tau)\mathrm{d}\tau, \tag{58}$$

with the $G(x, \xi, t-\tau)$ defined by (54). This is the same as that obtained by the Fourier method [Eq. (53)].

4. Concluding remarks

The principle of superposition decomposes the solution of hyperbolic heat conduction described by (2) into three parts $T_1(M, t)$, $T_2(M, t)$ and $T_3(M, t)$, representing the contribution of ψ, φ and f, respectively. Two theorems developed in this paper related the T_2 and T_3 to the T_1. This reduces the solution of (2) to that of a much simpler problem (4) and relates heat waves due to the initial temperature distribution φ and source disturbance f to that by the initial rate of temperature change ψ. The two theorems are further illustrated and justified through finding the solution of one-dimensional initial-boundary value problems with a homogeneous Dirichlet boundary condition.

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